Math137 - December 06, 2015

Qiu's Final Exam Review Package

Definitions you should know

§ **Limit**

Let f be a function defined on some open interval that contains the number **a**, except possibly at **a** itself.

$$\lim_{x\to a} f(\text{x}) = \infty$$

means that for every positive number M there is a positive number δ such that

$$\text{if } 0 < |\text{ x - a }| < \delta \text{ then } f(\text{x}) > M$$

L'Hospital's Rule

Suppose $\frac{f(x)}{g(x)}$ becomes either $\frac{0}{0}$ or $\pm\frac{\infty}{\infty}$ as $x \to a$, then

$$\lim_{x\to a} \frac{f(x)}{g(x)} = \lim_{x\to a} \frac{f'(x)}{g'(x)}$$

provided that the second limit EXISTS!

Squeeze Theorem

If $f(x) \leq g(x) \leq h(x)$ for all x near a (but not necessarily at x=a), and if $\lim f(x) = \lim h(x) = L$ then $\lim_{x\to a} g(x) = L$ as well

§ **Continuity**

A function f is **continuous at a number** a if

$$\lim_{x\to a} f(\text{x}) = f(\text{a})$$

Notice that this definition requires three things if f is continuous at a:

1. $f(a)$ is defined (that is, a is in the domain of f)
2. $\lim_{x\to a} f(x)$ exists
3. $\lim_{x\to a} f(x) = f(a)$

§ **Differentiability**

A function f is **differentiable** at a if *f'(a)* exists. It is **differentiable on an open interval** (a,b) [or (a, ∞) or (-∞ , a) or (- ∞ , ∞)] if it is differentiable at every number in the interval

Remark: if a function is differentiable then it is also continuous

The derivative of $f(x)$ is defined to be $f'(x)$
= [slope of tangent line to y = f(x) at (x, f(x))]

$$\text{Different ways you can find a derivative} \begin{cases} f'(x) = \lim_{t\to x} \frac{f(t)-f(x)}{t-x} \text{ (D1)} \\ f'(x) = \lim_{x\to h} \frac{f(x+h)-f(x)}{h} \text{ (D2)} \\ f'(a) = \lim_{x\to a} \frac{f(x)-f(a)}{x-a} \text{ (D3)} \end{cases}$$

Remark: Use the left + right limits when determining these

Also, differentiability can also be defined as:

f is differentiable at x if either of the previous limits exist and the limit is a finite real number.
E.g. f is differentiable at x = a $\leftrightarrow f'_+(a) = f'_-(a) =$ [finite number]

The logarithmic derivative

- $(log_a x)' = \frac{1}{x} * \frac{1}{lna}$ $(x \neq 0)$
- $(lnx)' = \frac{1}{x}$ where (x > 0)

Remark: Implicit diff. is similar to chain rule

§ **Hyperbolic Functions**
Special functions that have the same relationship with the hyperbola that the trig functions have with the circle (that is why they are named after the trig functions. e.g. hyperbolic sin → sinh)

§ sinhx = $\frac{e^x - e^{-x}}{2} \implies \frac{d}{dx}$ (sinhx) = $coshx$

§ coshx = $\frac{e^x + e^{-x}}{2} \implies \frac{d}{dx}$ (coshx) = $sinhx$

§ tanhx = $\frac{sinhx}{coshx} \implies \frac{d}{dx}$ (tanhx) = $sech^2x$

§ cschx = $\frac{1}{sinhx} \implies \frac{d}{dx}$ (cschx) = $-cschxcothx$

§ sechx = $\frac{1}{coshx} \implies \frac{d}{dx}$ (sechx) = $-sechxtanhx$

§ cothx = $\frac{coshx}{sinhx} \implies \frac{d}{dx}$ (cothx) = $-csch^2$x

§ **Max/Min points (local and global)**
Global
Let c be a number in the domain D of a function f. Then $f(c)$ is the

§ **absolute maximum** value of f on D if $f(c) \geq f(x)$ for all x in D

§ **absolute minimum** value of f on D if $f(c) \leq f(x)$ for all x in D

Remark: You can find the max/min by taking the boundary points and critical numbers (setting d/dy = 0) and subbing it into the original function and compare the results

Local
The number $f(c)$ is a

§ **local maximum** value of f if $f(c) \geq f(x)$ when x is near c

§ **local minimum** value of f if $f(c) \leq f(x)$ when x is near c

§ **Inflection Point**
A point P on a curve y = $f(c)$ is called an **inflection point** if f is continuous there and the curve changes from concave upward to concave downward or from concave downward to concave upward at P

§ **Increasing/ Decreasing Functions**

- A function f is called **increasing** on an interval I if

$$f(x_1) < f(x_2) \text{ whenever } x_1 < x_2 \text{ in } I$$

- It is called **decreasing** on I if

$$f(x_1) > f(x_2) \text{ whenever } x_1 < x_2 \text{ in } I$$

§ **Inverse Trig Functions**

Derivatives of Inverse Trig Functions

$$(sin^{-1}x)' = \frac{1}{\sqrt{1-x^2}} \qquad (cos^{-1}x)' = \frac{-1}{\sqrt{1-x^2}} \qquad (tan^{-1}x)' = \frac{1}{1+x^2}$$

Theorems or Methods you should memorize

§ **IVT**
Suppose that f is continuous on the closed interva [a,b] and let N be any number between $f(a)$ and $f(b)$ where $f(a) \neq f(b)$. Then there exists a number c in (a,b) such that $f(c) = N$

§ **MVT**
Rolle's Theorem: Let f be a function that satisfies the following three hypotheses:

1. f is continuous on the closed interval [a,b]
2. f is differentiable on the open interval (a,b)
3. $f(a) = f(b)$

Then there is a number c in (a,b) such that $f'(c) = 0$

The Mean Value Theorem: Let f be a function that satisfies the following hypotheses:

1. f is continuous on the closed interval [a,b]
2. f is differentiable on the open interval [a,b]

Then there is a number c in (a,b) such that

$$f'(c) = \frac{f(b)-f(a)}{b-a}$$

or, equivalently,

$$f(b) - f(a) = f'(c)(b-a)$$

§ **EVT**

The Extreme Value Theorem says that if f is continuous on a closed interval [a,b], then f attains an absolute value $f(c)$ and an absolute minimum value $f(d)$ at some numbers c and d in [a,b]

§ **Related Rates**

[Equation relating 2 quantities that change over time t]

$\Longrightarrow$ Take $\frac{d}{dt}$ of both sides using $\frac{d}{dt} = \frac{d\boxdot}{dt}\frac{d}{d\boxdot}$ (chain rule)

[New equation relating derivatives of 2 quantities]

If you are given one of the derivatives, then you can find the other derivative using the new equation

§ **Increasing/Decreasing Test**

1. If $f'(x)$>0 on an interval (a,b), then f is increasing on [a,b]
2. If $f'(x)$<0 on an interval (a,b), then f is decreasing on [a,b]

§ **Fundamental Theorem of Calculus Parts I and II**

Part I

If f is continuous on [a,b], then the function g defined by

$$g(x) = \int_a^x f(t)dt \quad a \leq x \leq b$$

is continuous on [a,b] and differential on (a,b) and $g'(x) = f(x)$

Part II

If f is continuous on [a,b], then

$$\int_a^b f(x)dx = F(b) - F(a)$$

where F is any antiderivative of f, that is, a function such that $F' = f$

§ **Newton's Method**

Definition: If f(n) = 0 , then we say x = r is a root of the eqn f(x) = 0

Ideas:

1. A root of f(x) = 0 is an x-intercept of y = f(x)
2. [x-int of y=f(x)] $\simeq$ [x-int of tangent line]

Then, we can build a sequence of real numbers such that $\lim_{n\to\infty} X_n$ = r

The tangent line at $(x_n, f(x_n))$ is given by y $-(f(x_n)) = f'(x_n)(x-x_n)$

$(x_{n+1}, 0)$ lies on this line so,

$$0 - f(x_n) = f'(x_n)(x_{n+1} - x_n)$$

$$\frac{-f(x_n)}{f'(x_n)} = x_{n+1} - x_n$$

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$$

§ **Linear Approximation**
Let f be diff. at x = a, then the function

$$L(x) = f(a) + f'(a)(x - a)$$

is called the linearization or linear approximation of f at x = a

§ **Curve Sketching**
"Guideline for sketching a curve" from the textbook

1. **Domain** Start with the set of values of x for which f(x) is defined
2. **Intercepts** The y-int is f(0) and this tells us where the curve intersects the y-axis. The x-ints happen when we solve f(x) = 0
3. **Symmetry**
   - § Even $\implies$ f(-x) = f(x) for all x in Domain
     Means that the curve is symmetric about the y-axis
   - § Odd $\implies$ f(-x) = -f(x) for all x in Domain
     Means that the curve is symmetric about the origin (think rotational symmetry)
   - § Periodic $\implies$ f(x+p) = f(x) for all x in Domain
     Means that the function repeats after p
4. **Asymptotes**
   - § Horizontal Asymptotes
     $$\lim_{x\to\infty} f(x) = L \text{ or } \lim_{x\to-\infty} f(x) = L$$
     then the line y = L is the H.A of the curve
   - § Vertical Asymptotes
     $$\lim_{x\to a^+} f(x) = \infty \qquad \lim_{x\to a^-} f(x) = \infty$$
     $$\lim_{x\to a^+} f(x) = -\infty \qquad \lim_{x\to a^-} f(x) = -\infty$$
     If any of the above are true, x = a is a vertical asymptote
5. **Intervals of Increase or Decrease**
   Use the I/D test and compute f'(x) and find the intervals on which f'(x) is positive (increasing) and where f'(x) is negative (decreasing)
6. **Local Max/Min Values**
   Find where f'(x) = 0 or DNE
   - § If f"(c) > 0 $\implies$ f(c) is a local min
   - § If f"(c) < 0 $\implies$ f(c) is a local max
7. **Concavity and Points of Inflection**
   Compute f"(x) = 0 to find where the direction of concavity changes

§ **Riemann Sums**
If f is integrable on [a,b] then

$$\int_a^b f(x)dx = \lim_{n\to\infty} \sum_{i=1}^{n} f(x_i)\Delta x$$

where

$$\Delta x = \frac{b-a}{n} \text{ and } x_i = a + i\Delta x$$

Midpoint Rule

$$\int_a^b f(x)dx \approx \sum_{i=1}^{n} f(\overline{x}_i)\Delta x = \Delta x[f(\overline{x}_1) + ... + f(\overline{x}_n)]$$

where

$$\Delta x = \frac{b-a}{n}$$

and

$$\overline{x}_i = \frac{1}{2}(x_{i-1} + x_i) = \text{ midpoint of } [x_{i-1}, xi]$$

§ **Area Between Curves**

The approximation is just a modified riemann sum

$$A = \lim_{n \to \infty} \sum_{i=1}^{n} [f(x_i^*) - g(x_i^*)]\Delta x$$

The area of the region bounded by the curves y = f(x) and y = g(x) over an interval [a,b] where f and g are continuous and f(x) $\geq$ g(x) for all x in [a,b] is

$$A = \int_a^b [f(x) - g(x)]dx$$

If there are moments when f(x) $\leq$ g(x) then we will have to split up the integrals whenever f(x) = g(x)
This turns the above equation into:

$$A = \int_a^b \mid f(x) - g(x) \mid dx$$